

On some estimators of the Hurst index of the solution of SDE driven by a fractional Brownian motion

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Abstract

Strongly consistent and asymptotically normal estimators of the Hurst parameter of solutions of stochastic differential equations are proposed. The estimators are based on discrete observations of the underlying processes.

Keywords: fractional Brownian motion, stochastic differential equation, Hurst index

1 Introduction

Recently long range dependence (LRD) became one of the most researched phenomena in statistics. It appears in various applied fields and inspires new models to account for it. Stochastic differential equations (SDEs) are widely used to model continuous time processes. Within this framework, LRD is frequently modeled with the help of SDEs driven by a fractional Brownian motion (fBm). It is well known that the latter Gaussian process is governed by a single parameter $H \in (0, 1)$ (called the Hurst index) and that values of H in $(1/2, 1)$ correspond to LRD models. In applications, the estimation of H is a fundamental problem. Its solution depends on the theoretical structure of a model under consideration. Therefore, particular models usually deserve separate analysis. In this paper, we concentrate on the estimation of H under the assumption that an observable continuous time process $(X_t)_{t \in [0, T]}$ satisfies SDE

$$X_t = \xi + \int_0^t f(X_s) ds + \int_0^t g(X_s) dB_s^H, \quad t \in [0, T], \quad (1.1)$$

where $T > 0$ is fixed, ξ is an initial r.v., f and g are continuous functions satisfying some regularity conditions, and $(B_t^H)_{t \in [0, T]}$ is a fBm with the Hurst index $1/2 < H < 1$. Our goal is to construct a strongly consistent and asymptotically normal estimator of the H from discrete observations X_{t_1}, \dots, X_{t_n} of trajectory X_t , $t \in [0, T]$.

We consider two cases. First, we examine the case when g is completely specified. Next, we relax this restriction and allow g to be unknown. Such situation may appear when g depends on additional nuisance parameters. In both cases, boundedness of $1/g$ plays an important role and is assumed to hold.

To our best knowledge, so far only several studies investigated this question. The pioneering work has been done by [3] of Berzin and León as well as lecture notes [4] with references therein. [12], [14] and [15] were also devoted to the problems of the same nature. However, all of these works focused on the strong consistency. The present paper is a generalization of [13] where a special case of (1.1) was considered.

The paper is organized in the following way. In Section 2 we present the main results of the paper. Section 3 is devoted to several results needed for the proofs. Sections 4–5 contain the proofs of the main results. Finally, in Section 6 two examples are given in order to illustrate the obtained results.

2 Main results

To avoid cumbersome expressions, we introduce symbols O_ω, o_ω . Let (Y_n) be a sequence of r.v., ς is an a.s. non-negative r.v. and $(a_n) \subset (0, \infty)$ vanishes. $Y_n = O_\omega(a_n)$ means that $|Y_n| \leq \varsigma \cdot a_n$; $Y_n = o_\omega(a_n)$ means that $|Y_n| \leq \varsigma \cdot b_n$ with $b_n = o(a_n)$. In particular, $Y_n = o_\omega(1)$ corresponds to the sequence (Y_n) which tends to 0 a.s. as $n \rightarrow \infty$.

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†This research was funded by a grant (No. MIP-048/2014) from the Research Council of Lithuania.

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Let $\pi_n = \{\tau_k^n, k = 0, \dots, i_n\}$, $n \geq 1$, $\mathbb{N} \ni i_n \uparrow \infty$, be a sequence of partitions of the interval $[0, T]$. If partition π_n is uniform then $\tau_k^n = \frac{kT}{i_n}$ for all $k \in \{0, \dots, i_n\}$. If $i_n \equiv n$, we write t_k^n instead of τ_k^n . In order to formulate our main results, we state two hypotheses:

$$\begin{aligned} (\mathbf{H}) \quad & \Delta X_{\tau_k^n} = X_{\tau_k^n} - X_{\tau_{k-1}^n} = O_\omega(d_n^{H-\varepsilon}), \quad k = 1, \dots, i_n; \\ (\mathbf{H}_1) \quad & \Delta^{(2)} X_{\tau_k^n} = X_{\tau_k^n} - 2X_{\tau_{k-1}^n} + X_{\tau_{k-2}^n} = g(X_{\tau_{k-1}^n}) \Delta^{(2)} B_{\tau_k^n}^H + O_\omega(d_n^{2(H-\varepsilon)}), \\ & k = 2, \dots, i_n, \end{aligned} \quad (2.1)$$

for all $\varepsilon \in (0, H - \frac{1}{2})$, where $d_n = \max_{1 \leq k \leq i_n} (\tau_k^n - \tau_{k-1}^n)$.

THEOREM 2.1. *Assume that solution of Eq. (1.1) satisfies hypothesis (\mathbf{H}_1) . Moreover, let the function g is known, Lipschitz-continuous and there exists a random variable ς such that $P(\varsigma < \infty) = 1$ and*

$$\sup_{t \in [0, T]} \frac{1}{|g(X_t)|} \leq \varsigma \quad a.s. \quad (2.2)$$

Then

$$\begin{aligned} \hat{H}_n^{(1)} &\longrightarrow H \quad a.s., \\ 2\sqrt{n} \ln \frac{n}{T} (\hat{H}_n^{(1)} - H) &\xrightarrow{d} N(0; \sigma_H^2) \quad \text{for } H \in (1/2, 1), \end{aligned}$$

where σ_H^2 is a known variance defined in Subsection 3.1,

$$\hat{H}_n^{(1)} = \varphi_{n,T}^{-1} \left(\frac{1}{n} \sum_{i=2}^n \left(\frac{\Delta^{(2)} X_{t_i^n}}{g(X(t_{i-1}^n))} \right)^2 \right) \quad \text{for } n > T,$$

$$\varphi_{n,T}(x) = \left(\frac{T}{n} \right)^{2x} (4 - 2^{2x}) \quad \text{and } \varphi_{n,T}^{-1} \text{ denotes an inverse of } \varphi_{n,T}, \quad x \in (0, 1), \quad n > T. \quad \square$$

It is natural to try to drop restriction of the known g . For this purpose we need several additional definitions. Assume that the process X is observed at time points $\frac{i}{m_n}T$, $i = 1, \dots, m_n$, where $m_n = nk_n$, and k_n grows faster than $n \ln n$, but the growth does not exceed polynomial, e.g. $k_n = n \ln^\theta n$, $\theta > 1$, or $k_n = n^2$.

Denote

$$W_{n,k} = \sum_{j=-k_n+2}^{k_n} \left(\Delta^{(2)} X_{s_j^n + t_k^n} \right) = \sum_{j=-k_n+2}^{k_n} \left(X_{s_j^n + t_k^n} - 2X_{s_{j-1}^n + t_k^n} + X_{s_{j-2}^n + t_k^n} \right)^2,$$

where $1 \leq k \leq n-1$ and $s_j^n = \frac{j}{m_n}T$.

THEOREM 2.2. *Assume that solution of Eq. (1.1) satisfies hypotheses (\mathbf{H}) and (\mathbf{H}_1) . Moreover, let the function g is Lipschitz-continuous and there exists a random variable ς such that $P(\varsigma < \infty) = 1$ and inequality (2.2) holds. Then*

$$\begin{aligned} \hat{H}_n^{(2)} &\longrightarrow H \quad a.s., \\ 2\sqrt{n} \ln \frac{n}{T} (\hat{H}_n^{(2)} - H) &\xrightarrow{d} N(0; \sigma_H^2) \quad \text{for } H \in (1/2, 1), \end{aligned}$$

where

$$\hat{H}_n^{(2)} = \frac{1}{2} + \frac{1}{2 \ln k_n} \ln \left(\frac{2}{n} \sum_{k=2}^n \frac{(\Delta^{(2)} X_{t_k^n})^2}{W_{n,k-1}} \right),$$

and σ_H^2 is a known variance defined in Subsection 3.1.

3 Preliminaries

3.1 Several results on fBm

Recall that fBm $(B_t^H)_{t \geq 0}$ with the Hurst index $H \in (0, 1)$ is a real-valued continuous centered Gaussian process with covariance given by

$$\mathbf{E}(B_t^H B_s^H) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).$$

For consideration of strong consistency and asymptotic normality of the given estimators we need several facts regarding B^H .

Limit results. Let

$$V_{n,T} = \frac{n^{2H-1}}{T^{2H}(4 - 2^{2H})} \sum_{k=2}^n (\Delta^{(2)} B_{t_k^n}^H)^2, \quad H \neq \frac{1}{2}.$$

Then (see [2], [10], [1])

$$V_{n,T} \xrightarrow[n \rightarrow \infty]{} 1 \quad \text{a.s.},$$

$$\sqrt{n}(V_{n,T} - 1) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2),$$

where

$$\sigma_H^2 = 2 \left(1 + 2 \sum_{j=1}^{\infty} \rho_H^2(j) \right), \quad \rho_H(j) = - \frac{|j-2|^{2H} - 4|j-1|^{2H} + 6|j|^{2H} - 4|j+1|^{2H} + |j+2|^{2H}}{2(4 - 2^{2H})}.$$

Hölder-continuity of B^H . It is known that almost all sample paths of an fBm B^H are locally Hölder of order strictly less than H , $H \in (0, 1)$. To be more precise, for all $0 < \varepsilon < H$ and $T > 0$ there exists a nonnegative random variable $G_{\varepsilon, T}$ such that $\mathbb{E}(|G_{\varepsilon, T}|^p) < \infty$ for all $p \geq 1$, and

$$|B_t^H - B_s^H| \leq G_{\varepsilon, T} |t - s|^{H-\varepsilon} \quad \text{a.s.} \quad (3.1)$$

for all $s, t \in [0, T]$.

3.2 Concentration inequality

Let

$$Y_{k,n} = \frac{n^H}{T^H \sqrt{4 - 2^{2H}}} \Delta^{(2)} B_{t_k}^H, \quad d_{jk}^{(2)n} = \mathbf{E} Y_{j,n} Y_{k,n}, \quad j, k = 2, \dots, n.$$

Note that $d_{jk}^{(2)n} = \rho_H(j-k)$. In the sequel we make use of the following modified version of an inequality of concentration from [6].

LEMMA 3.1. *For all $z > 0$ and any $H \in (0, 1)$,*

$$\mathbf{P} \left(\left| (n-1)^{-1/2} \sum_{k=2}^n (Y_{k,n}^2 - 1) \right| > z \right) \leq 2 \exp \left(- \frac{z^2}{\frac{32}{3} \left(\frac{z}{\sqrt{n-1}} + 1 \right)} \right).$$

Proof. Let $\varkappa = \sup_{H \in (0,1)} \sum_{j \in \mathbb{Z}} |\rho_H(j)|$. Following an argument of the paper [6], one gets bound

$$\begin{aligned} 2(n-1)^{-1} \sum_{k,j=2}^n Y_{k,n} Y_{j,n} d_{kj}^{(2)n} &\leq 2(n-1)^{-1} \sum_{k,j=2}^n |Y_{k,n}| \cdot |Y_{j,n}| \cdot |\rho_H(j-k)| \\ &\leq 2(n-1)^{-1} \sum_{k,j=2}^n Y_{k,n}^2 |\rho_H(j-k)| \\ &\leq 2(n-1)^{-1} \varkappa \sum_{k=2}^n Y_{k,n}^2 = 2(n-1)^{-1} \varkappa \sum_{k=2}^n (Y_{k,n}^2 - 1) + 2\varkappa \\ &= \frac{2\varkappa}{\sqrt{n-1}} \left(\frac{1}{\sqrt{n-1}} \sum_{k=2}^n (Y_{k,n}^2 - 1) \right) + 2\varkappa. \end{aligned}$$

Thus (see [6]),

$$\mathbf{P} \left(\left| \frac{1}{\sqrt{n-1}} \sum_{k=2}^n (Y_{k,n}^2 - 1) \right| > z \right) \leq 2 \exp \left\{ - \frac{z^2}{4\varkappa \left(\frac{z}{\sqrt{n-1}} + 1 \right)} \right\}.$$

In paper [5] it was proved that

$$\sum_{j \in \mathbb{Z}} |\rho_H(j)| = \begin{cases} 1 + \frac{10 - 7 \cdot 2^{2H} + 2 \cdot 3^{2H}}{4 - 2^{2H}} & \text{for } H \leq 1/2, \\ 1 + \frac{4 - 2^{2H}}{4 - 2^{2H}} = 2, & \text{for } H \geq 1/2, \end{cases}$$

and

$$\varkappa = \sup_{H \in (0,1)} \sum_{j \in \mathbb{Z}} |\rho_H(j)| = \lim_{H \rightarrow 0+} \sum_{j \in \mathbb{Z}} |\rho_H(j)| = \frac{8}{3}.$$

This yields the required inequality.

4 Proof of Theorem 2.1

Proof of Theorem 2.1. Observe first that

$$\varphi_{n,T}(x) = \left(\frac{T}{n}\right)^{2x} (4 - 2^{2x}), \quad x \in (0, 1),$$

is continuous and strictly decreasing for $n > T$. Thus, it has an inverse $\varphi_{n,T}^{-1}$ for $n > T$. By hypothesis **(H₁)**,

$$\begin{aligned} \frac{\varphi_{n,T}(\hat{H}_n^{(1)})}{\varphi_{n,T}(H)} &= \left[\left(\frac{T}{n}\right)^{2H} (4 - 2^{2H})\right]^{-1} \varphi_{n,T} \left(\varphi_{n,T}^{-1} \left(\frac{1}{n} \sum_{i=2}^n \left(\frac{\Delta^{(2)} X_{t_i^n}}{g(X_{t_{i-1}^n})} \right)^2 \right) \right) \\ &= \left[\left(\frac{T}{n}\right)^{2H} (4 - 2^{2H})\right]^{-1} \left(\frac{1}{n} \sum_{i=2}^n \left(\frac{\Delta^{(2)} X_{t_i^n}}{g(X_{t_{i-1}^n})} \right)^2 \right) \\ &= \frac{n^{2H-1}}{T^{2H} (4 - 2^{2H})} \left(\sum_{i=2}^n \left(\Delta^{(2)} B_{t_i^n}^H \right)^2 + O_\omega \left(\frac{1}{n^{3(H-\varepsilon)-1}} \right) \right) \\ &= \frac{n^{2H-1}}{T^{2H} (4 - 2^{2H})} \sum_{i=2}^n \left(\Delta^{(2)} B_{t_i^n}^H \right)^2 + O_\omega \left(\frac{1}{n^{H-3\varepsilon}} \right) = V_{n,T} + O_\omega \left(\frac{1}{n^{H-3\varepsilon}} \right) \end{aligned}$$

for $3\varepsilon < H$. Therefore

$$\frac{\varphi_{n,T}(\hat{H}_n^{(1)})}{\varphi_{n,T}(H)} \longrightarrow 1 \quad \text{a.s. as } n \rightarrow \infty.$$

Using the same arguments as in [13] it is possible to prove that the estimator $\hat{H}_n^{(1)}$ is strongly consistent and asymptotically normal.

5 Proof of the main Theorem

Before presenting the proof of this theorem, we give two auxiliary lemmas.

LEMMA 5.1. *Let*

$$V_{n,T}(k) = \frac{m_n^{2H}}{2k_n T^{2H} (4 - 2^{2H})} \sum_{j=-k_n+2}^{k_n} \left(B_{s_j^n + t_k^n}^H - 2B_{s_{j-1}^n + t_k^n}^H + B_{s_{j-2}^n + t_k^n}^H \right)^2, \quad 1 \leq k \leq n-1.$$

The following relation holds:

$$\max_{1 \leq k \leq n-1} |V_{n,T}(k) - 1| = O_\omega \left(\sqrt{\frac{\ln n}{k_n}} \right).$$

Proof. By self similarity and stationarity of increments of fBm,

$$\begin{aligned} V_{n,T}(k) &\stackrel{d}{=} \frac{2^{2H-1} m_n^{2H}}{2^{2H} k_n (4 - 2^{2H})} \sum_{j=-k_n+2}^{k_n} \left(B_{\frac{j+k_n}{m_n}}^H - 2B_{\frac{j+k_n-1}{m_n}}^H + B_{\frac{j+k_n-2}{m_n}}^H \right)^2 \\ &\stackrel{d}{=} \frac{(2m_n)^{2H-1} n}{(2n)^{2H} (4 - 2^{2H})} \sum_{j=-k_n+2}^{k_n} \left(B_{\frac{j}{k_n}+1}^H - 2B_{\frac{j-1}{k_n}+1}^H + B_{\frac{j-2}{k_n}+1}^H \right)^2 \\ &\stackrel{d}{=} \frac{(2k_n)^{2H-1}}{2^{2H} (4 - 2^{2H})} \sum_{j=2}^{2k_n} \left(B_{\frac{j}{k_n}}^H - 2B_{\frac{j-1}{k_n}}^H + B_{\frac{j-2}{k_n}}^H \right)^2 \\ &\stackrel{d}{=} \frac{(2k_n)^{2H-1}}{4 - 2^{2H}} \sum_{j=2}^{2k_n} \left(B_{\frac{j}{2k_n}}^H - 2B_{\frac{j-1}{2k_n}}^H + B_{\frac{j-2}{2k_n}}^H \right)^2 \stackrel{d}{=} V_{2k_n,1}. \end{aligned}$$

Therefore,

$$\mathbb{P} \left(\max_{1 \leq k \leq n-1} |V_{n,T}(k) - 1| > \delta \right) \leq \sum_{k=1}^{n-1} \mathbb{P} (|V_{n,T}(k) - 1| > \delta) \leq n \mathbb{P} (|V_{2k_n,1} - 1| > \delta) \quad \text{for all } \delta > 0.$$

Put $\hat{V}_{n,T} = \frac{n}{n-1} V_{n,T}$. Note that

$$|V_{2k_n,1} - 1| \leq |\hat{V}_{2k_n,1} - 1| + \frac{1}{2k_n}.$$

Let (δ_n) be a sequence of positive numbers such that $\delta_n \downarrow 0$ as $n \rightarrow \infty$ and $k_n^{-1} < \delta_n$. By Lemma 3.1,

$$\begin{aligned} \mathbb{P}(|V_{2k_n,1} - 1| > 2\delta_n) &\leq \mathbb{P}\left(|\widehat{V}_{2k_n,1} - 1| + \frac{1}{2k_n} > 2\delta_n\right) \leq \mathbb{P}\left(|\widehat{V}_{2k_n,1} - 1| > \delta_n\right) \\ &= \mathbb{P}\left(\sqrt{2k_n - 1} |\widehat{V}_{2k_n,1} - 1| > \delta_n \sqrt{2k_n - 1}\right) \leq 2 \exp\left\{-\frac{\delta_n^2(2k_n - 1)}{\frac{32}{3}(\delta_n + 1)}\right\}. \end{aligned}$$

Set $\delta_n = \sqrt{a \frac{\ln n}{2k_n - 1}}$. Since $k_n \geq n \ln n$, then

$$\mathbb{P}(|V_{2k_n,1} - 1| > 2\delta_n) \leq 2 \exp\left\{-\frac{3a \ln n}{32 \left(\sqrt{a \frac{\ln n}{2k_n - 1}} + 1\right)}\right\}.$$

If $a \geq 3$ and $n \geq 2$, then $\delta_n > k_n^{-1}$. Moreover, $\mathbb{P}(|V_{2k_n,1} - 1| > 2\delta_n) \leq 2n^{-3}$ for $a > 32$ and n large enough. Therefore series $\sum_n \mathbb{P}(\max_{1 \leq k \leq n-1} |V_{n,T}(k) - 1| > \delta_n)$ converges and by the Borel-Cantelli lemma $\max_{1 \leq k \leq n-1} |V_{n,T}(k) - 1| \xrightarrow[n \rightarrow \infty]{} 0$ a.s.

LEMMA 5.2. Assume that function g is Lipschitz-continuous. If $\varepsilon < (H - 1/2)/3$, then for each $k = 1, \dots, n - 1$

$$W_{n,k} = g^2(X_{t_k^n}) \frac{T^{2H}(4 - 2^{2H})}{(2k_n)^{2H-1}} + O_\omega\left(\frac{\sqrt{k_n \ln n}}{m_n^{2H}}\right) + O_\omega\left(\frac{k_n}{n^{H-\varepsilon} m_n^{2(H-\varepsilon)}}\right).$$

Proof. Step 1. By hypothesis (\mathbf{H}_1) ,

$$\Delta^{(2)} X_{s_j^n + t_k^n} = g(X_{s_{j-1}^n + t_k^n}) \Delta^{(2)} B_{s_j^n + t_k^n}^H + O_\omega\left(\frac{1}{m_n^{2(H-\varepsilon)}}\right), \quad j = -k_n + 2, \dots, k_n.$$

Next, note that

$$\begin{aligned} |g^2(X_t) - g^2(X_s)| &= |g(X_t) - g(X_s)| |g(X_t) + g(X_s)| \leq L |X_t - X_s| |g(X_s) + g(X_t)| \\ &\leq 2L \sup_{u \in [0, T]} |g(X_u)| |X_t - X_s| \end{aligned}$$

where $t > s$ and L is Lipschitz constant. Thus, hypothesis (\mathbf{H}) with a.s. continuity of $t \mapsto g(X_t)$ lead to

$$g^2(X_{s_{j-1}^n + t_k^n}) - g^2(X_{t_k^n}) = O_\omega\left(\frac{1}{n^{H-\varepsilon}}\right).$$

Step 2. Assume that $\varepsilon < (H - 1/2)/3$. *Step 1* and Lemma 5.1 yield

$$\begin{aligned} W_{n,k} &= \sum_{j=-k_n+2}^{k_n} \left(\Delta^{(2)} X_{s_j^n + t_k^n}\right)^2 = \sum_{j=-k_n+2}^{k_n} g^2(X_{s_{j-1}^n + t_k^n}) \left(\Delta^{(2)} B_{s_j^n + t_k^n}^H\right)^2 + O_\omega\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ &= g^2(X_{t_k^n}) \sum_{j=-k_n+2}^{k_n} \left(\Delta^{(2)} B_{s_j^n + t_k^n}^H\right)^2 + \sum_{j=-k_n+2}^{k_n} \left(g^2(X_{s_{j-1}^n + t_k^n}) - g^2(X_{t_k^n})\right) \left(\Delta^{(2)} B_{s_j^n + t_k^n}^H\right)^2 \\ &\quad + O_\omega\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ &= g^2(X_{t_k^n}) \frac{2k_n T^{2H}(4 - 2^{2H})}{m_n^{2H}} V_{n,T}(k) + O_\omega\left(\frac{k_n}{n^{H-\varepsilon} m_n^{2(H-\varepsilon)}}\right) + O_\omega\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ &= g^2(X_{t_k^n}) \frac{2k_n T^{2H}(4 - 2^{2H})}{m_n^{2H}} + O_\omega\left(\frac{\sqrt{k_n \ln n}}{m_n^{2H}}\right) + O_\omega\left(\frac{k_n}{n^{H-\varepsilon} m_n^{2(H-\varepsilon)}}\right). \end{aligned}$$

Consequently the proof of lemma is completed.

Proof of Theorem 2.2. Put

$$S_{n,T} := \frac{2}{nk_n^{2H-1}} \sum_{k=2}^n \frac{(\Delta^{(2)} X_{t_k^n})^2}{W_{n,k-1}}.$$

It follows from (2.1)–(2.2) and Lemma 5.2 that

$$\begin{aligned}
S_{n,T} &= \frac{2}{nk_n^{2H-1}} \sum_{k=2}^n \frac{g^2(X_{t_k^n}) (\Delta^{(2)} B_{t_k^n}^H)^2 + O_\omega(n^{-3(H-\varepsilon)})}{g^2(X_{t_k^n}) \frac{2k_n T^{2H}(4-2^{2H})}{m_n^{2H}} + O_\omega\left(\frac{\sqrt{k_n \ln n}}{m_n^{2H}}\right) + O_\omega\left(\frac{k_n}{n^{H-\varepsilon} m_n^{2(H-\varepsilon)}}\right)} \\
&= \frac{2}{nk_n^{2H-1}} \sum_{k=2}^n \frac{(\Delta^{(2)} B_{t_k^n}^H)^2 + O_\omega(n^{-3(H-\varepsilon)})}{\frac{2k_n T^{2H}(4-2^{2H})}{m_n^{2H}} + O_\omega\left(\frac{\sqrt{k_n \ln n}}{m_n^{2H}}\right) + O_\omega\left(\frac{k_n}{n^{H-\varepsilon} m_n^{2(H-\varepsilon)}}\right)} \\
&= \frac{m_n^{2H}}{nk_n^{2H} T^{2H}(4-2^{2H})} \sum_{k=2}^n \frac{(\Delta^{(2)} B_{t_k^n}^H)^2 + O_\omega(n^{-3(H-\varepsilon)})}{1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)} \\
&= \frac{V_{n,T} + O_\omega(n^{-(H-3\varepsilon)})}{1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)}.
\end{aligned}$$

Term $O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)$ vanishes provided $\varepsilon > 0$ is small enough. Convergence $V_{n,T} \xrightarrow{a.s.} 1$ implies that $S_{n,T} \xrightarrow{a.s.} 1$. Hence

$$H_n^{(2)} = H + \frac{\ln S_{n,T}}{2 \ln k_n} \xrightarrow[n \rightarrow \infty]{a.s.} H.$$

To prove asymptotic normality of the estimator $H_n^{(2)}$ observe that

$$\begin{aligned}
\sqrt{n}(S_{n,T} - 1) &= \sqrt{n} \left(\frac{V_{n,T} - 1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right) + O_\omega(n^{-(H-3\varepsilon)})}{1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)} \right) \\
&= \frac{\sqrt{n}(V_{n,T} - 1)}{1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)} \\
&\quad + \frac{O_\omega\left(\sqrt{\frac{n \ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-1/2-\varepsilon}}\right) + O_\omega(n^{-(H-1/2-3\varepsilon)})}{1 + O_\omega\left(\sqrt{\frac{\ln n}{k_n}}\right) + O_\omega\left(\frac{m_n^{2\varepsilon}}{n^{H-\varepsilon}}\right)} \xrightarrow{d} N(0, \sigma_H^2)
\end{aligned}$$

for $\varepsilon > 0$ small enough. Now apply Slutsky's theorem and limit results of Section 3.1.

6 Examples

As mentioned previously, in this section we present two examples of applications of the obtained results. The first one deals with a general form of equation (1.1) and relies on certain restrictions on functions f and g . The second one describes a particular model which formally does not fit into the scope of the first one.

6.1 Example 1

In order to present an example, we need several facts on variation. To make the paper more self-contained and the structure clearer, the mentioned facts are briefly reminded in subsection 6.1.1. For details we refer the reader to [9].

6.1.1 Variation

Fix $p > 0$ and $-\infty < a < b < \infty$. Let $\varkappa = \{\{x_0, \dots, x_n\} \mid a = x_0 < \dots < x_n = b, n \geq 1\}$ denotes a set of all possible partitions of $[a, b]$. For any $f : [a, b] \rightarrow \mathbb{R}$ define

$$v_p(f; [a, b]) = \sup_{\varkappa} \sum_{k=1}^n |f(x_k) - f(x_{k-1})|^p, \quad V_p(f; [a, b]) = v_p^{1/p}(f; [a, b]),$$

$$\mathcal{W}_p([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} \mid v_p(f; [a, b]) < \infty\}, \quad \mathcal{CW}_p([a, b]) = \{f \in \mathcal{W}_p([a, b]) \mid f \text{ is continuous}\}.$$

Recall that v_p is called p -variation of f on $[a, b]$ and any f in $\mathcal{W}_p([a, b])$ is said to have bounded p -variation on $[a, b]$. For short we omit an interval $[a, b]$ in the notations introduced above whenever there is no ambiguity. Below we list several facts used further on.

- $f \mapsto V_p(f)$ is a seminorm on \mathcal{W}_p ; $V_p(f) = 0$ if and only if f is a constant.
- $f \in \mathcal{W}_p \Rightarrow \sup_{x \in [a, b]} |f(x)| < \infty$.

- $f, g \in \mathcal{W}_p \Rightarrow fg \in \mathcal{W}_p$.
- $q > p \geq 1 \Rightarrow \mathcal{W}_p \subset \mathcal{W}_q$.
- Let $f \in \mathcal{W}_q$, $h \in \mathcal{W}_p$ with $p, q \in (0, \infty)$ such that $1/p + 1/q > 1$. Then an integral $\int_a^b f dh$ exists as the Riemann–Stieltjes integral provided f and h have no common discontinuities. If the integral exists, the Love–Young inequality

$$\left| \int_a^b f dh - f(y)[h(b) - h(a)] \right| \leq C_{p,q} V_q(f) V_p(h)$$

holds for all $y \in [a, b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n \geq 1} n^{-s}$. Moreover,

$$V_p \left(\int_a^{\cdot} f dh; [a, b] \right) \leq C_{p,q} V_{q,\infty}(f) V_p(h),$$

where $V_{q,\infty}(f) = V_q(f) + \sup_{x \in [a,b]} |f(x)|$. Also note that $V_{q,\infty}$ is a norm on \mathcal{W}_q , $q \geq 1$.

REMARK 6.1. The left-hand side of (3.1) can be replaced by $V_{H_\varepsilon}(B^H; [s, t])$, $\forall \varepsilon \in (0, H - \frac{1}{2})$, i.e.

$$V_{H_\varepsilon}(B^H; [s, t]) \leq G_{\varepsilon,T} |t - s|^{H-\varepsilon} \text{ a.s.} \quad (6.1)$$

with $G_{\varepsilon,T}$ of (3.1).

6.1.2 Assumptions and properties of solution of SDE

Let $(B_t^H)_{t \in [0,T]}$, $H \in (1/2, 1)$, be a fixed fBm defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F})$. Let $\alpha \in (\frac{1}{H} - 1; 1]$, $\mathcal{C}^{1+\alpha}(\mathbb{R}) = \{h : \mathbb{R} \rightarrow \mathbb{R} \mid h' \text{ exists and } \sup_x |h'(x)| + \sup_{x \neq y} \frac{|h'(x) - h'(y)|}{|x - y|^\alpha} < \infty\}$. Assume that f is Lipschitz and $g \in \mathcal{C}^{1+\alpha}(\mathbb{R})$. In such case there exists a unique solution of (1.1) having the following properties: i) $(X_t)_{t \in [0,T]}$ is \mathbb{F} adapted and almost all sample paths are continuous; ii) $X_0 = \xi$ a.s.; iii) $\forall p > \frac{1}{H}$, $\mathbf{P}(V_p(X; [0, T]) < \infty) = 1$ (see [8], [16], [17] and [11]).

LEMMA 6.1. Let X satisfies (1.1), $\varepsilon \in (0, H - \frac{1}{2})$. There exists a.s. finite r.v. $L_{\varepsilon,T}$ such that

$$V_{H_\varepsilon}(X; [s, t]) \leq L_{\varepsilon,T} (t - s)^{(H-\varepsilon)}, \quad 0 \leq s < t \leq T. \quad (6.2)$$

Proof. Since V_p is seminorm and non-increasing function of $p \geq 1$, inequalities of subsection 6.1.1 give bound

$$\begin{aligned} V_{H_\varepsilon}(X; [s, t]) &\leq \int_s^t |f(X_u)| du + V_p \left(\int_s^{\cdot} g(X_u) dB_u^H; [s, t] \right) \\ &\leq \sup_{u \in [0,T]} |f(X_u)| (t - s) + C_{H_\varepsilon, H_\varepsilon} \left(V_{H_\varepsilon}(g \circ X; [s, t]) + \sup_{u \in [0,T]} |g(X_u)| \right) V_{H_\varepsilon}(B^H; [s, t]) \\ &\leq L_{\varepsilon,T} (t - s)^{H-\varepsilon} \end{aligned}$$

with $L_{\varepsilon,T} = \sup_{u \in [0,T]} |f(X_u)| T^{1-H+\varepsilon} + C_{H_\varepsilon, H_\varepsilon} \left(V_{H_\varepsilon}(g \circ X; [s, t]) + \sup_{u \in [0,T]} |g(X_u)| \right) G_{\varepsilon,T}$ and $G_{\varepsilon,T}$ of (6.1). A.s. continuity of $t \mapsto X_t$ together with continuity of f and g implice a.s. boundedness of $\sup_{u \in [0,T]} |f(X_u)|$ and $\sup_{u \in [0,T]} |g(X_u)|$. It is not difficult to show that $V_{H_\varepsilon}(g \circ X; [s, t]) \leq \sup_u |g'(u)| V_{H_\varepsilon}(X; [s, t])$. Hence, continuity of g' and a.s. boundedness of $V_{H_\varepsilon}(X; [s, t])$ guarantees a.s. boundedness of $V_{H_\varepsilon}(g \circ X; [s, t])$ along with that of $L_{\varepsilon,T}$.

LEMMA 6.2. Let X satisfies (1.1), $\varepsilon \in (0, H - \frac{1}{2})$, and conditions stated above are true. Then the following relations hold:

$$\Delta X_{\tau_k^n} = O_\omega(d_n^{H-\varepsilon}), \quad k = 1, \dots, i_n, \quad (6.3)$$

$$\Delta^{(2)} X_{\tau_k^n} = g(X_{\tau_{k-1}^n}) \Delta^{(2)} B_{\tau_k}^H + O_\omega(d_n^{2(H-\varepsilon)}), \quad k = 2, \dots, i_n, \quad (6.4)$$

where $d_n = \max_{1 \leq k \leq i_n} (\tau_k^n - \tau_{k-1}^n)$.

Proof. Let a sample path $t \mapsto X_t$ be continuous. We first prove (6.3). Note that

$$\Delta X_{\tau_k^n} = X_{\tau_k^n} - X_{\tau_{k-1}^n} = \int_{\tau_{k-1}^n}^{\tau_k^n} f(X_s) ds + \int_{\tau_{k-1}^n}^{\tau_k^n} [g(X_s) - g(X_{\tau_{k-1}^n})] dB_s^H + g(X_{\tau_{k-1}^n}) \Delta B_{\tau_k}^H.$$

An application of inequalities (6.1)–(6.2) together with continuity of f, g, g' and mean value theorem yield

$$\begin{aligned}
\int_{\tau_{k-1}^n}^{\tau_k^n} |f(X_s)| ds &\leq \sup_{u \in [0, T]} |f(X_u)| (\tau_k^n - \tau_{k-1}^n) = O_\omega(d_n), \\
\left| \int_{\tau_{k-1}^n}^{\tau_k^n} [g(X_s) - g(X_{\tau_{k-1}^n})] dB_s^H \right| &\leq \|g'\|_\infty C_{H_\varepsilon, H_\varepsilon} V_{H_\varepsilon}(X; [\tau_{k-1}^n, \tau_k^n]) V_{H_\varepsilon}(B^H; [\tau_{k-1}^n, \tau_k^n]) \\
&= O_\omega(d_n^{2(H-\varepsilon)}), \\
|g(X_{\tau_k^n}) \Delta B_{\tau_k^n}^H| &\leq \sup_{u \in [0, T]} |g(X_u)| G_{\varepsilon, T} (\tau_k^n - \tau_{k-1}^n)^{H-\varepsilon} = O_\omega(d_n^{H-\varepsilon}).
\end{aligned} \tag{6.5}$$

Therefore (6.3) holds.

Next we prove (6.4). Since

$$\Delta^{(2)} X_{\tau_k^n} = \int_{\tau_{k-1}^n}^{\tau_k^n} f(X_s) ds - \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} f(X_s) ds + \int_{\tau_{k-1}^n}^{\tau_k^n} g(X_s) dB_s^H - \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} g(X_s) dB_s^H,$$

and

$$\begin{aligned}
&\int_{\tau_{k-1}^n}^{\tau_k^n} g(X_s) dB_s^H - \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} g(X_s) dB_s^H = \int_{\tau_{k-1}^n}^{\tau_k^n} [g(X_s) - g(X_{\tau_{k-1}^n})] dB_s^H \\
&\quad - \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} [g(X_s) - g(X_{\tau_{k-1}^n})] dB_s^H + g(X_{\tau_{k-1}^n}) (\Delta B_{\tau_k^n}^H - \Delta B_{\tau_{k-1}^n}^H) \\
&\stackrel{(6.5)}{=} O_\omega(d_n^{2(H-\varepsilon)}) + g(X_{\tau_{k-1}^n}) \Delta^{(2)} B_{\tau_k^n}^H, \quad k = 2, \dots, i_n,
\end{aligned}$$

then by mean value theorem with K equal to Lipschitz constant of f ,

$$\begin{aligned}
\left| \int_{\tau_{k-1}^n}^{\tau_k^n} f(X_s) ds - \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} f(X_s) ds \right| &\leq \int_{\tau_{k-1}^n}^{\tau_k^n} |f(X_s) - f(X_{\tau_{k-1}^n})| ds + \int_{\tau_{k-2}^n}^{\tau_{k-1}^n} |f(X_{\tau_{k-1}^n}) - f(X_s)| ds \\
&\leq 2K d_n \max_{1 \leq k \leq i_n} \sup_{\tau_{k-1}^n \leq s \leq \tau_k^n} |X_s - X_{\tau_{k-1}^n}| \stackrel{(6.3)}{=} O_\omega(d_n^{1+H-\varepsilon}).
\end{aligned}$$

We conclude that (6.4) also holds.

6.1.3 Application

Results of subsection 6.1.2 imply proposition constituting the basis of the first example.

PROPOSITION 6.1. *Assume that a model defined by (1.1) satisfies conditions of subsection 6.1.2. If in addition (2.2) holds, then Theorems 2.1–2.2 apply.*

6.2 Example 2

Consider the Verhulst equation

$$X_t = \xi + \int_0^t (\lambda X_s - X_s^2) dt + \sigma \int_0^t X_s dB_t^H, \quad \xi > 0, \quad t \in [0, T].$$

$f(x) = \lambda x - x^2$ is not Lipschitz. So we can't use results of section 6.1. It was proved in [13] that this equation admits an explicit solution

$$X_t = \frac{\xi \exp\{\lambda t + \sigma B_t^H\}}{1 + \xi \int_0^t \exp\{\lambda s + \sigma B_s^H\} ds}, \quad t \in [0, T].$$

Moreover, from [13] one can get

$$\begin{aligned}
\Delta X_{t_k^n} &= X_{t_{k-1}^n} \left(\lambda \frac{T}{n} + \sigma \Delta B_{t_k^n}^H + O_\omega\left(\frac{1}{n}\right) \right), \quad k = 1, \dots, n, \\
\Delta^{(2)} X_{t_k^n} &= X_{t_{k-1}^n} \left(\sigma \Delta^{(2)} B_{t_k^n}^H + O_\omega\left(\frac{1}{n^{2(H-\varepsilon)}}\right) \right), \quad k = 2, \dots, n.
\end{aligned}$$

Finally, explicit form of X_t implies existence of an a.s. finite r.v. ς such that $\sup_{t \in [0, T]} \frac{1}{|X_t|} \leq \varsigma$ a.s. Thus, one can use Theorem 2.1 if constant σ is known and Theorem 2.2 in general situation.

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